# Crop Circle Theorems

## Their Proofs and Relationship to Musical Notes

This research began with a simple and rather limited objective: to prove the crop circle theorems of Dr. Gerald Hawkins. In fact if I could have found the proofs in the literature of the field, this research would never have taken place at all. Fortunately, I couldn't find them because once I started, I could see further work that needed to be done.

As I proved Dr. Hawkins theorems, I discovered five new ones and proved them as well. I then took the diatonic ratios of all the theorems and related them to the frequencies of the musical scale. With some rather startling results I might add.

Beginning with **Theorem IA** I need to make some observations that apply to all of the theorems. In Euclidean Geometry one almost always has to see the end before making a beginning. Also, since we are looking for diatonic ratios, we need to find an equation or equations which will let us divide one diameter or radius by the other. Remember too, that because we are working with ratios, the constants divide out leaving diameter ratios equal to radius ratios. And if we square them they are equal to each other and to the ratio of the areas.

Applying this to Theorem IA the equation we need to write is for the diameter of the circumscribing circle. It contains both the radii of the initial and the circumscribed circle. So from the equation we are able to divide it and find the diatonic ratio of 4 to 3.

Although, I have proved three more Theorem I's, I believe this is the one Dr Hawkins meant when he said Theorem I. See Circular Relationships for The Theorems in Appendix A. I base this belief on the 4 to 3 diatonic relationship which is related directly to Note F above Middle C. See Frequencies In The Fields in Appendix B.

**Theorem IB** is like Theorem IA except that the equilateral triangle is inscribed rather than being circumscribed. It can be proved by Theorem IA and Theorem II. The equations already exist so just divide them for the proof. This gives a new diatonic ratio which is also the Note F, one octave lower than the previous.
This theorem is such a simple and logical extension of the first two that I am puzzled as to why Dr Hawkins did not discover and publish it.

**Theorem IC** is also often referred to as Theorem I although it is quite different. Sometimes both Theorem IA and Theorem IC appear in the same article as if they were identical. They aren’t. The proof of Theorem IC shows that it contains no diatonic ratio that can be related to a musical frequency. I believe that this was not the theorem Dr. Hawkins was referring to when he said Theorem I. In my mind the origin of Theorem IC is rather murky.

**Theorem ID** would have never been discovered if I had read the instructions for constructing Theorem IA a little closer. Instead of circumscribing the equilateral triangle, I circumscribed the three circles and then proved the theorem before realizing my mistake. It has a nice 7 to 3 relationship but it would need to be 8 to 3 to be Note F in the next higher octave.

**Theorem II** is easy to prove by constructing the appropriate similar triangles and remembering their relationships. It may be proved a number of different ways I have shown just one of them. It has the nice diatonic ratio of 4 to 1 which relates directly to the Note C which is two octaves above Middle C.

**Theorem III** is the simplest of all proofs. Just remember the Pythagorean Theorem. It also has a nice diatonic ratio of 2 to 1 which relates directly to the Note C which is an octave above Middle C.

Again using the Pythagorean Theorem, **Theorem IVA** is shown to have a nice 4 to 3 relationship. We have previously related this to Note F using Theorem IA.

While proving Theorem II, it occurred to me that there should be a similar theorem related to the hexagon. There was and that led me to discover **Theorem IVB** by connecting the diameters at the hexagon corners. Again by using similar triangles and writing and dividing the proper equations it is shown to have a diatonic ratio of 1 to 3 which relates directly to the Note F. This Note F is yet another octave lower.

I have included **Theorem IVC** mostly for completeness as it does not have a diatonic ratio which can be related to a specific note. If I hadn’t included it you might have wondered why since it can be proved by simply dividing Theorem IVB by Theorem IVA.

There is a **Theorem V** which can be used for deriving (not proving) the other theorems. However it does not of its self have diatonic ratios and therefore was not a part of this research.

Appendix A **Circular Relationships for The Theorems** shows a summary of all the results. Note that to go from one column to the other, you simply square or take the square root. But how do you know which column to use? I have followed the lead of Dr. Hawkins in that if the circles are not concentric, you use the ratio of diameters, if they are concentric you use the ratio of areas. This means diameters for Theorems IA, IB, IC, ID, IVB, and IVC and areas for Theorems II, III, and IVA. Why did he pick this convention? Certainly I don't know, perhaps he was a practical man and he did it *because it works*.

**Frequencies In The Fields** in Appendix B gives four octaves: two above and two below Middle C. This does not encompass the full 27.5 to 4,186 Hz of a piano but does include all the frequencies found so far. Notice that all the notes are either F or C. Coincidence or a message? Perhaps as we discover more notes, this will become clear.

**Theorem T** in Appendix C is not really a part of this research, but is included as help for anyone wanting to compute circle and regular polygon ratios. It includes all cases and relies on trigonometry rather than Euclidian Geometry.

Finally, if you're wondering about me, I have a **Short Bio** in Appendix D.

# Theorem IA

If three equal circles are tangent to a common line and their centers can be connected by an equilateral triangle and a circle is circumscribed about the triangle, the ratio of the diameters is 4 to 3.

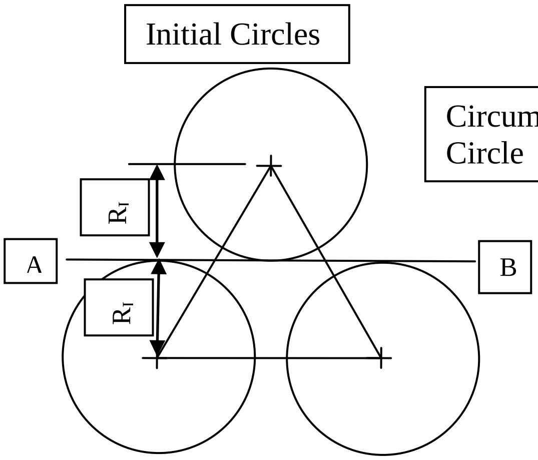

Figure 1. Circles with equilateral triangle.

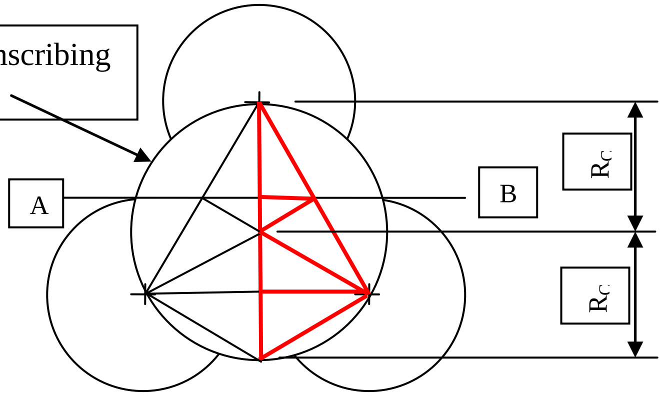

Figure 2. Circumscribed and bisected equilateral triangle.

Proof:

1. In Triangles CDE and CFG, Side ED is twice the length of Side GF by similar triangles.
2. In Triangles FGH and DEH, $\frac{2d}{d} = \frac{f}{e}$ therefore f = 2e. By symmetry HE = EJ.
3. In circumscribed circle, Diameter ($D_C$) = 2 * Radius ($R_C$) = CH and HJ
4. In the initial circles, ($D_C$) = CG + GE + EJ = $R_I + R_I + \frac{2R_I}{3}$
5. $2R_C = 2R_I + \frac{2R_I}{3}$. Therefore $\frac{R_C}{R_I} = \frac{4}{3}$
6. $2R_I = D_I$, $2R_C = D_C$, Therefore $\frac{D_C}{D_I} = \frac{4}{3}$ **Proving the Theorem.**
7. Further, $\frac{Area_C}{Area_I} = \frac{D_C^2}{D_I^2} = \frac{R_C^2}{R_I^2} = \frac{16}{9}$ .

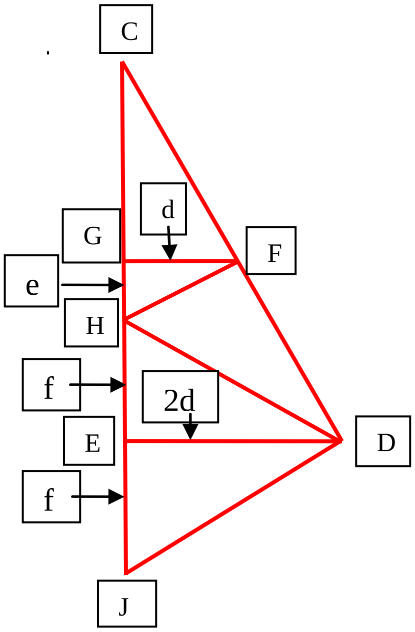

Figure 3. Enlarged section for clarity of proof

Notes: There are two versions of Theorem I in the literature, Internet, books, etc. This is the version that appears most often. For that reason and the fact that it has the nice 4:3 relationship, I believe that this is the one Dr. Hawkins meant when he said Theorem I.

# Theorem IB

If three equal circles are tangent to a common line and their centers can be connected by an equilateral triangle and a circle is inscribed within the triangle, the ratio of the diameters is 2:3.

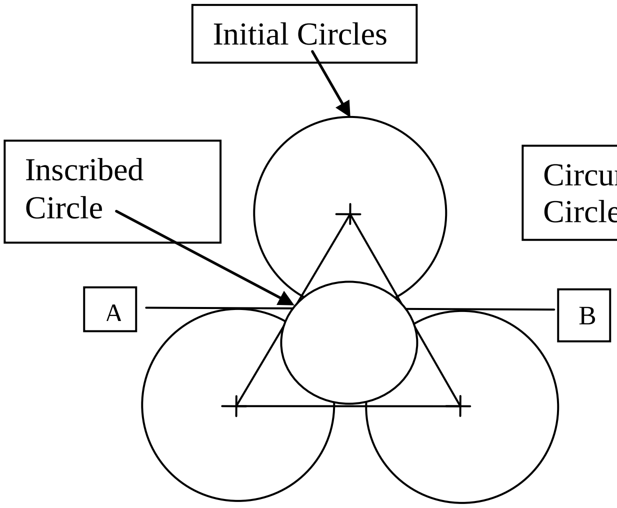

Figure 1. Circles with inscribed equilateral triangle.

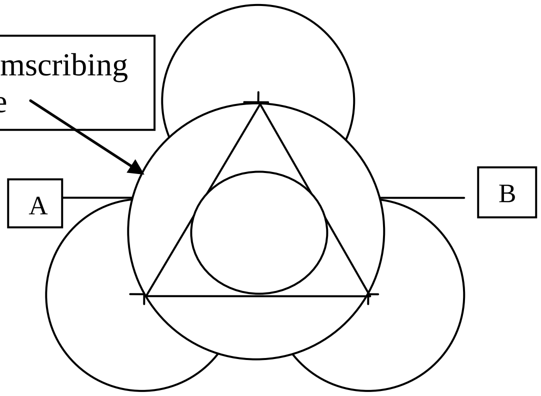

Figure 2. Circumscribed and inscribed equilateral triangle.

Proof:

1. The ratio of diameters between the Circumscribing Circle ($D_C$) and the Initial Circles ($D_I$) is 4:3. Per Theorem IA.
2. That is $\frac{D_C}{D_I} = \frac{4}{3}$
3. The ratio of areas between circles inscribed and circumscribed about an equilateral triangle is 4:1 Per Theorem II
4. That is $\frac{Area_C}{Area_{IN}} = \frac{4}{1}$ and $\frac{Area_C}{Area_{IN}} = \frac{D_C^2}{D_{IN}^2}$ so taking the square root $\frac{D_C}{D_{IN}} = 2$
5. And $\frac{D_{IN}}{D_C} = \frac{1}{2}$
6. So Multiplying $\frac{D_{IN}}{D_C}$ by $\frac{D_C}{D_I} = \frac{D_{IN}}{D_I} = \frac{2}{3}$
7. Thereby, **<u>Proving the Theorem</u>**.

8. Further: $\frac{Area_{IN}}{Area_I} = \frac{D_{IN}^2}{D_I^2} = \frac{R_{IN}^2}{R_I^2} = \frac{4}{9}$

Notes: This theorem does not appear in the literature, Internet, books, etc. However its simplicity and its logical progression make it puzzling as to why it was not published by Dr. Hawkins.

# Theorem IC

If three equal circles are tangent to a common line and their centers can be connected by an equilateral triangle and a circle is constructed using the single circle as a center and drawing the circle through the other two centers, the ratio of the diameters is 4 to Sqrt 3.

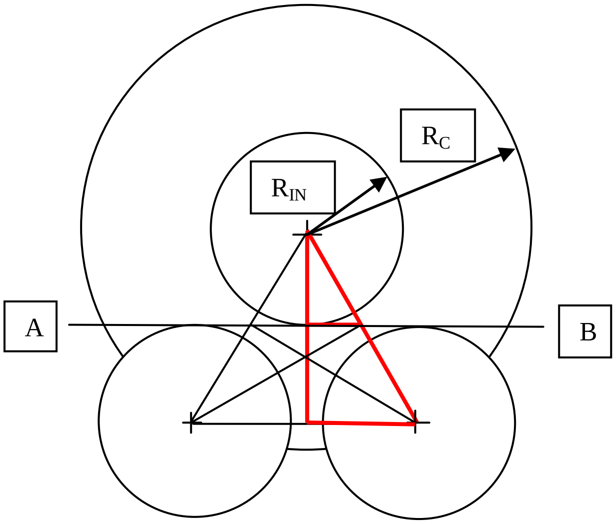

Figure 1. Circles with bisected equilateral triangle.

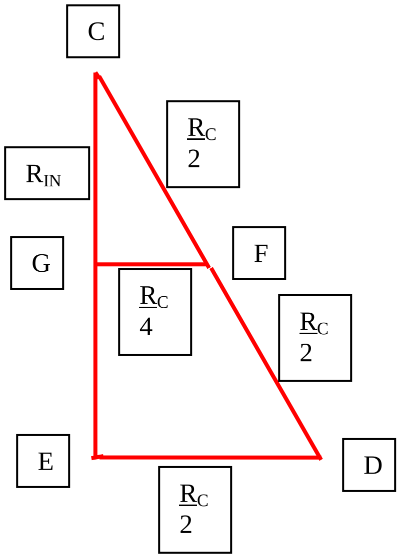

Figure 2. Enlarged section for clarity of proof

Proof:

1. In Triangles CDE and CFG, Side ED ($\frac{R_C}{2}$) is twice the length of Side GF ($\frac{R_C}{4}$) by similar triangles.
2. Side DE ($\frac{R_C}{2}$) is equal to Side CF ($\frac{R_C}{2}$) by congruent triangles. See Figure 1.
3. In Triangle CFG, $\frac{R_C^2}{(2)^2} = R_{IN}^2 + \frac{R_C^2}{(4)^2}$ by the Pythagorean Theorem
4. That is, $\frac{R_C^2}{4} = R_{IN}^2 + \frac{R_C^2}{16}$ and $\frac{R_C^2}{4} - \frac{R_C^2}{16} = R_{IN}^2$
5. Giving, $R_{IN}^2 = \frac{3\,R_C^2}{16}$ So that, $\frac{R_C^2}{R_{IN}^2} = \frac{D_C^2}{D_{IN}^2} = \frac{\text{Area}_C}{\text{Area}_{IN}} = \frac{16}{3}$.
6. Taking the square root, $\frac{D_C}{D_{IN}} = \frac{4}{\text{Sqrt } 3}$ **Proving the Theorem.**

Notes: There are two versions of Theorem I in the literature, Internet, books, etc. This version appears in some. It does not have the nice 4:3 integer relationship shown in Theorem IA. I believe that this is probably not the one Dr. Hawkins meant when he said Theorem I. It is not clear what the origin of this theorem might be.

# Theorem ID

If three equal circles are tangent to a common line and their centers can be connected by an equilateral triangle and a circle is constructed circumscribing the three circles, the ratio of the diameters is 7 to 3.

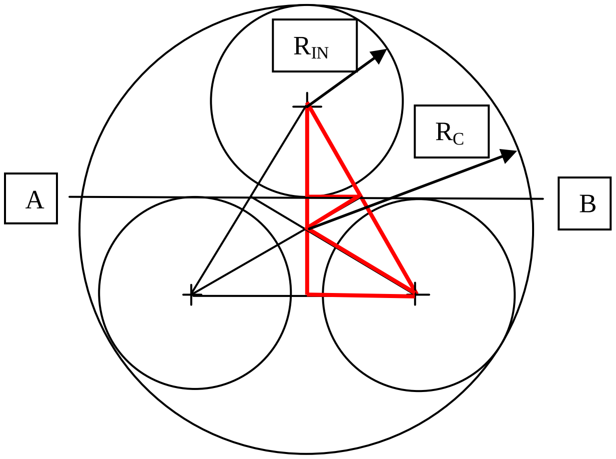

Figure 1. Circles with bisected equilateral triangle.

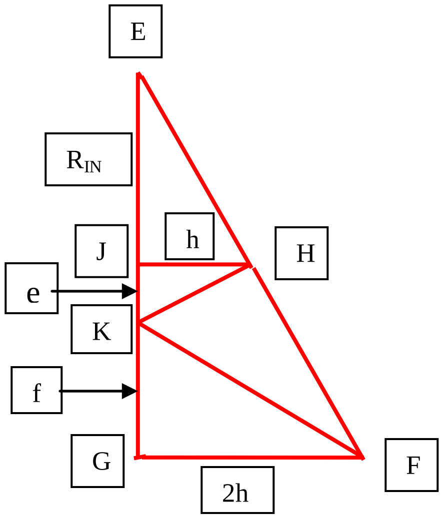

Figure 2. Enlarged section for clarity of proof

Proof:

1. In Triangles EFG and EHJ, Side GF (2h) is twice the length of Side JH (h) by similar triangles.
2. In Triangles JHK and KFG, $\frac{e}{h} = \frac{f}{2h}$ So, $e = \frac{f\,h}{2h} = \frac{f}{2}$ Or $f = 2e$
3. $R_{IN} = e + f = 3e$, Giving, $e = \frac{R_{IN}}{3}$
4. $R_C = 2\,R_{IN} + e = 2\,R_{IN} + \frac{R_{IN}}{3}$ Giving, $R_C = \frac{7\,R_{IN}}{3}$
5. So, $\frac{R_C}{R_{IN}} = \frac{D_C}{D_{IN}} = \frac{7}{3}$ **Proving the Theorem.**
6. Furthermore, $\frac{Area_C}{Area_{IN}} = \frac{(D_C)^2}{(D_{IN})^2} = \frac{(R_C)^2}{(R_{IN})^2} = \frac{49}{9}$.

Notes: This version of Theorem I does not appear in the literature, Internet, books, etc. In fact it would not appear here except for my initial misunderstanding of the instructions for constructing Theorem I. I circumscribed the three circles instead of the equilateral triangle. However it does have a nice 7:3 integer relationship. So, I am including it along with the others.

# Theorem II

If an equilateral triangle is inscribed and circumscribed the ratio of the circles' areas is 4:1.

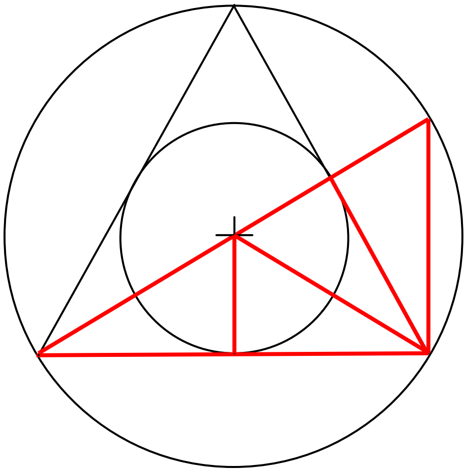

Figure 1. Circumscribed and inscribed equilateral triangle.

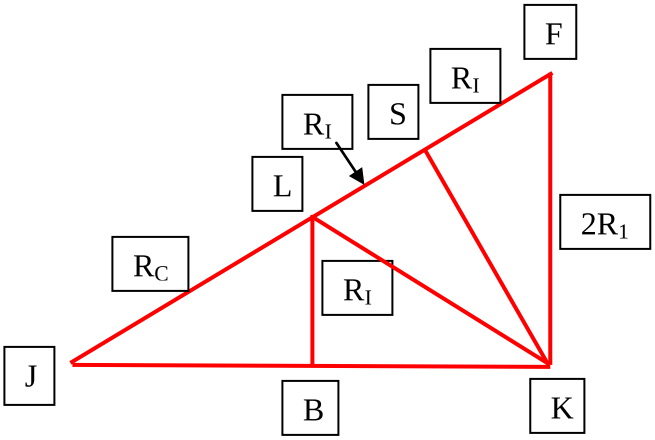

Figure 2. Removed and enlarged section for clarity of proof

Proof:

There are about 10 ways to prove this theorem. This is one of them.

1. In Triangles JFK and LBJ, Side FK ($2R_1$) is twice the length of Side LB ($R_1$) by similar triangles.
2. In Triangle LFK, the angles are all $60^0$, therefore it is an equilateral triangle and all legs are the same length.
3. So, Side SF is also $R_I$ in length.
4. In Diameter JF, $R_C = R_I + R_I = 2R_I$
5. Giving $\frac{R_C}{R_I} = \frac{D_C}{D_I} = 2$
6. So, $\frac{Area_C}{Area_I} = \frac{(D_C)^2}{(D_I)^2} = \frac{(R_C)^2}{(R_I)^2} = 4$ **Proving the Theorem.**

Trigonometric Verification:

Angle JLB is $60^0$. So, $\cos 60^0 = \frac{R_I}{R_C} = 0.50$, and $\frac{R_c}{R_I} = 2$

and $\frac{(R_C)^2}{(R_I)^2} = 4$

Notes: Euclidian geometry is useful for proving the concentric circular relationships of polygons for only the equilateral triangle, the square, and the hexagon. However, all regular polygons of any number of sides may be proved by using only one simple trigonometric equation. The theorem, equation, an illustrative figure, the proof, and a table of the most common polygons are included in my Theorem T. If you would like a copy, send me an E-mail at deegragg@yahoo.com and ask for Theorem T.

# Theorem III

If a square is inscribed and circumscribed the ratio of the circles' areas is 2:1.

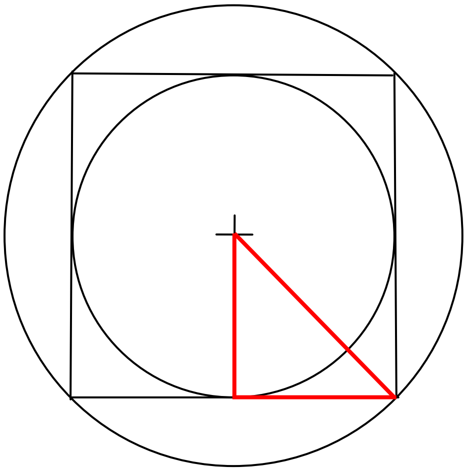

Figure 1. Circumscribed and inscribed square.

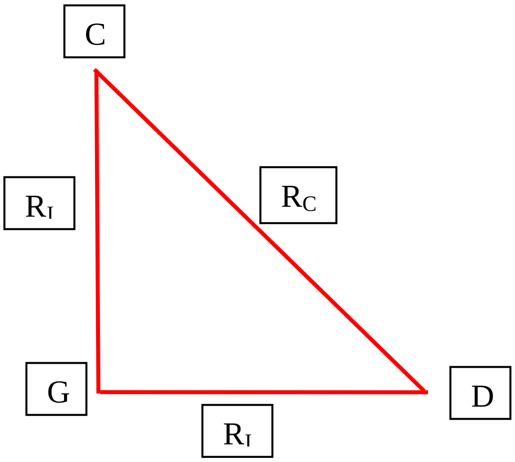

Figure 2. Removed and enlarged section for clarity of proof

Proof:

1. Triangle CDG is a $45^0$ isosceles triangle, therefore Side CG = Side GD = $R_I$

2. So, $(R_C)^2 = (R_I)^2 + (R_I)^2$ per the Pythagorean Theorem.

3. So, $(R_C)^2 = 2(R_I)^2$, giving $\frac{(R_C)^2}{(R_I)^2} = \frac{Area_C}{Area_I} = 2$ **Proving the Theorem.**

Trigonometric Verification:

Angle GCD is $45^0$. So, $\cos 45^0 = \frac{R_I}{R_C} = 0.707$, and $\frac{R_c}{R_I} = 1.414$

and $\frac{(R_C)^2}{(R_I)^2} = 2$

Notes: Euclidian geometry is useful for proving the concentric circular relationships of polygons for only the equilateral triangle, the square, and the hexagon. However, all regular polygons of any number of sides may be proved by using only one simple trigonometric equation. The theorem, equation, an illustrative figure, the proof, and a table of the most common polygons are included in my Theorem T. If you would like a copy, send me an E-mail at deegragg@yahoo.com and ask for Theorem T.

# Theorem IVA

If a hexagon is inscribed and circumscribed the ratio of the circles' areas is 4:3.

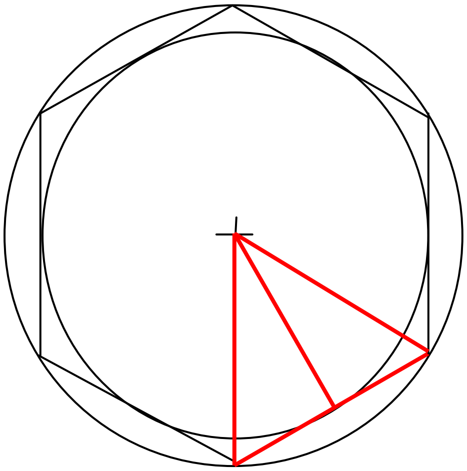

Figure 1. Circumscribed and inscribed hexagon.

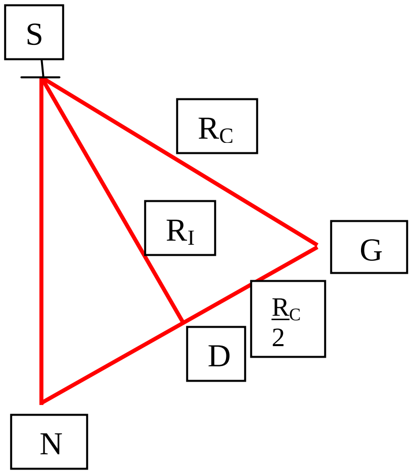

Figure 2. Removed and enlarged section for clarity of proof

Proof:

1. Triangle NSG is an equilateral triangle. Therefore Leg SG = Leg NG.
2. Line SD is perpendicular to and bisects Line NG. So, Line DG = $\frac{R_C}{2}$
3. So, in Triangle SDG $(R_C)^2 = (R_I)^2 + \frac{(R_C)^2}{(2)^2}$ per the Pythagorean Theorem
4. Giving $(R_I)^2 = (R_C)^2 - \frac{(R_C)^2}{4} = \frac{3(R_C)^2}{4}$
5. So, $\frac{(R_C)^2}{(R_I)^2} = \frac{(D_C)^2}{(D_I)^2} = \frac{Area_C}{Area_I} = \frac{4}{3}$ **Proving the Theorem.**
6. Further, $\frac{R_C}{R_I} = \frac{D_C}{D_I} = \frac{2}{Sqrt3}$

Trigonometric Verification:
Angle GSD is $30^0$. So, $\cos 30^0 = \frac{R_I}{R_C} = 0.866$, and $\frac{R_c}{R_I} = 1.1547 = \frac{2}{Sqrt3}$

Notes: Euclidian geometry is useful for proving the concentric circular relationships of polygons for only the equilateral triangle, the square, and the hexagon. However, all regular polygons of any number of sides may be proved by using only one simple trigonometric equation. The theorem, equation, an illustrative figure, the proof, and a table of the most common polygons are included in my Theorem T. If you would like a copy, send me an E-mail at deegragg@yahoo.com and ask for Theorem T.

# Theorem IVB

If a hexagon is inscribed and circumscribed and the corners connected by diameters, the inscribed circles of the created equilateral triangles have a diameter ratio to the inscribed circle of 1:3.

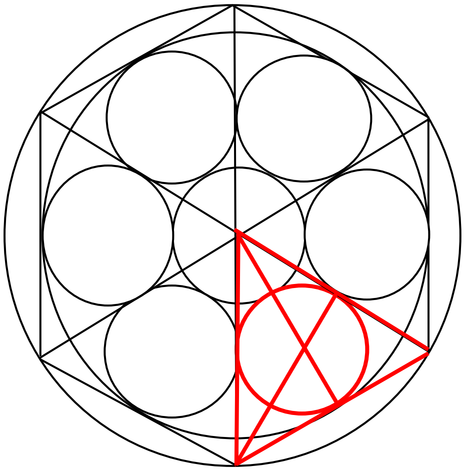

Figure 1. Circumscribed and inscribed hexagon with connecting diameters.

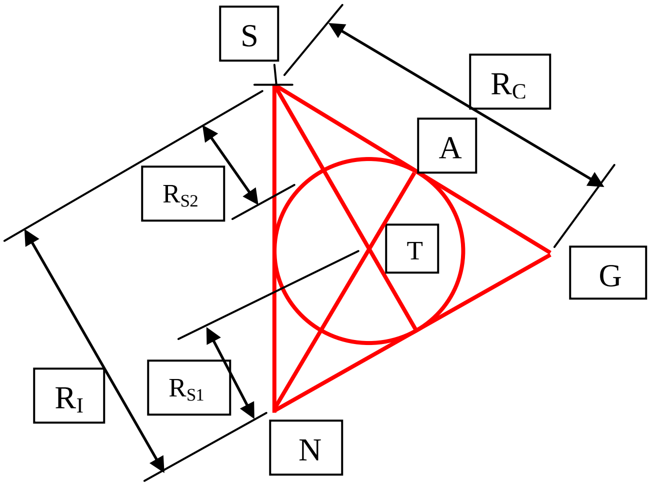

Figure 2. Removed and enlarged section for clarity of proof

Proof:

1. $R_I = 2R_{S1} + R_{S2}$.

2. In Triangle NAG, $NG = R_C$, and $AG = \frac{R_C}{2}$ per equilateral triangles.

3. In Triangles SAT and NAG, $\frac{R_{S1}}{R_{S1} + R_{S2}} = \frac{\frac{R_C}{2}}{R_C} = \frac{1}{2}$ by similar triangles.

4. Giving $R_{S1} = \frac{R_{S1}}{2} + \frac{R_{S2}}{2}$

5. So, $2R_{S1} = R_{S1} + R_{S2}$ and $R_{S1} = R_{S2}$

6. From Line 1. $R_I = 2R_{S1} + R_{S2} = 3R_{S1}$

7. So, $\frac{R_{S1}}{R_I} = \frac{D_{S1}}{D_I} = \frac{1}{3}$ **Proving the Theorem.**

Further, $\frac{(R_{S1})^2}{(R_I)^2} = \frac{(D_{S1})^2}{(D_I)^2} = \frac{Area_{S1}}{Area_I} = \frac{1}{9}$

# Theorem IVC

If a hexagon is inscribed and circumscribed and the corners connected by diameters, the inscribed circles of the created equilateral triangles have a diameter ratio to the circumscribed circle of 1:2Sqrt3.

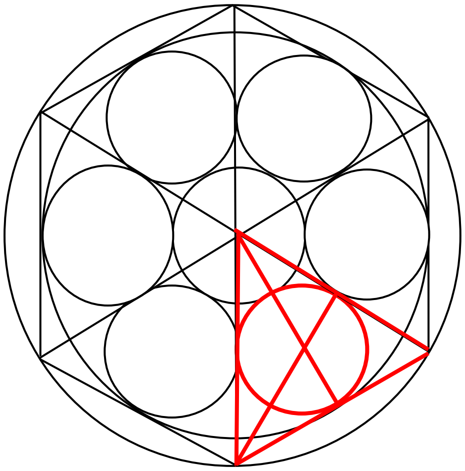

Figure 1. Circumscribed and inscribed hexagon with connecting diameters.

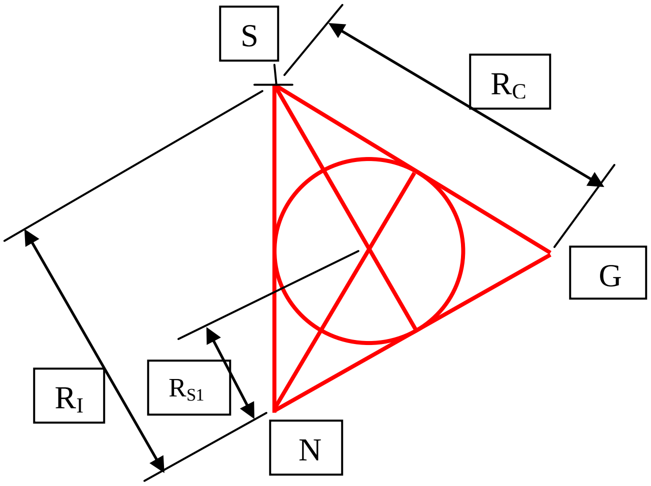

Figure 2. Removed and enlarged section for clarity of proof

Proof:

1. $\frac{R_{SI}}{R_I} = \frac{1}{3}$ Per Theorem IVB
2. $\frac{R_C}{R_I} = \frac{2}{\text{Sqrt } 3}$ Per Theorem IVA
3. Dividing the equations $\frac{R_{SI}}{R_C} = \frac{\frac{1}{3}}{\frac{2}{\text{Sqrt } 3}} = \frac{1}{3} \text{ X } \frac{\text{Sqrt } 3}{2}$
4. So, $\frac{R_{SI}}{R_C} = \frac{D_{SI}}{D_C} = \frac{1}{2 \text{ Sqrt } 3}$ **Proving the Theorem.**

Further, $\frac{(R_{S1})^2}{(R_C)^2} = \frac{(D_{S1})^2}{(D_C)^2} = \frac{\text{Area}_{S1}}{\text{Area}_C} = \frac{1}{12}$

# Appendix A

## Circular Relationships for The Theorems

| Theorem | Ratio of Diameters and Radii | Ratio of Areas, Diameters Squared, and Radii Squared |
|---|---|---|
| Theorem IA | 4:3 | 16:9 |
| Theorem IB | 2:3 | 4:9 |
| Theorem IC | 4:Sqrt3 | 16:3 |
| Theorem ID | 7:3 | 49:9 |
| Theorem II | 2:1 | 4:1 |
| Theorem III | Sqrt2:1 | 2:1 |
| Theorem **IVA** | 2:Sqrt3 | 4:3 |
| Theorem **IVB** | 1:3 | 1:9 |
| Theorem **IVC** | 1:2 Sqrt3 | 1:12 |
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# Appendix B

## Frequencies In The Fields

| Note Name | C | D | E | F | G | A | B | C |
|---|---|---|---|---|---|---|---|---|
| Diatonic Ratio | 1/4 | 9/32 | 5/16 | $\boxed{1/3}$ | 3/8 | 5/12 | 15/32 | 1/2 |
| Frequency (Hz) | 66 | 74.25 | 82.5 | 88 | 99 | 110 | 123.75 | 132 |
| | | | | | | | | |
| Note Name | C | D | E | F | G | A | B | C |
| Diatonic Ratio | 1/2 | 9/16 | 5/8 | $\boxed{2/3}$ | 3/4 | 5/6 | 15/16 | 1 |
| Frequency (Hz) | 132 | 148.5 | 165 | 176 | 198 | 220 | 247.5 | 264 |
| | | | | | | | | |
| Note Name | C* | D | E | F | G | A | B | C |
| Diatonic Ratio | 1 | 9/8 | 5/4 | $\boxed{4/3}$ | 3/2 | 5/3 | 15/8 | $\boxed{2}$ |
| Frequency (Hz) | 264 | 297 | 330 | 352 | 396 | 440 | 495 | 528 |
| | | | | | | | | |
| Note Name | C | D | E | F | G | A | B | C |
| Diatonic Ratio | 2 | 9/4 | 5/2 | 8/3 | 3 | 10/3 | 15/4 | $\boxed{4}$ |
| Frequency (Hz) | 528 | 594 | 660 | 704 | 792 | 880 | 990 | 1056 |

* Middle C

$\boxed{\quad}$ Denotes found in the fields

## Theorem Summary

| Frequency (Hz) | Theorem Used For Proof |
|---|---|
| 88 | Theorem IVB, Gragg |
| 176 | Theorem IB, Gragg |
| 352 | Theorem IA, Theorem IVA, Hawkins |
| 528 | Theorem III, Hawkins |
| 1056 | Theorem II, Hawkins |

# Appendix C
# Theorem T

Trigonometry can be used to solve circular relationships for inscribed and circumscribed regular polygons for polygons of any number of sides from 3 to infinity.

Proof:

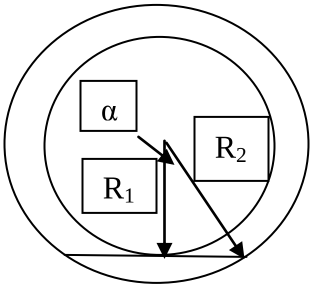

Figure 1. Regular polygon with any number of sides

Where: $\alpha = \frac{360^0}{2n}$ and n = number of sides

$$\cos \alpha = \frac{R_1}{R_2}$$

$$\frac{R_2}{R_1} = \frac{1}{\cos \alpha}$$ **Proving the Theorem**

Further: $$\frac{(R_2)^2}{(R_1)^2} = \frac{(1)^2}{(\cos \alpha)^2}$$

**Table of Some Common Polygons**

| Figure (All are equiangular) | Number of Equal Sides | Ratio of Diameters and Radii | Ratio of Areas, Diameters Squared, and Radii Squared |
|---|---|---|---|
| Triangle (1)(4) | 3 | 2.000 | 4 4.000 |
| Square (2) | 4 | 1.414 | 2 2.000 |
| Pentagon | 5 | 1.236 | 1.527 |
| Hexagon (3) | 6 | 1.155 | 4/3 1.333 |
| Heptagon | 7 | 1.110 | 1.232 |
| Octagon (4) | 8 | 1.082 | 1.172 |
| Nonagon | 9 | 1.064 | 1.132 |
| Decagon | 10 | 1.051 | 1.106 |
|  | 15 | 1.022 | 1.045 |
|  | 20 | 1.012 | 1.025 |
|  | 50 | 1.002 | 1.004 |
|  | 100 | 1.000 | 1.001 |
|  | 200 | 1.000 | 1.000 |
|  | ∞ | 1.000 | 1.000 |

(1) Theorem II, by Dr. Hawkins using Euclidian Geometry
(2) Theorem III, by Dr. Hawkins using Euclidian Geometry

(3) Theorem IV, by Dr. Hawkins using Euclidian Geometry
(4) Found in the Kekoskee/Mayville, Wisconsin Crop Circle
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# Appendix D

## Short Bio

My name is Dee Gragg. I am a retired, mechanical engineer. My career was spent in research, testing and evaluation. My main areas of research were automotive air bags, jet aircraft ejection seats and high speed rocket sleds. I have published 33 technical papers as either the principal author or a co-author. They form a part of the body of literature in their respective fields.